MATH LEAGUE
BAG OF TRICKS

Finding Prime Factors Of Large Numbers

See if a number can be rewritten using algebraic factoring rules.

Difference of squares: $x^2 - y^2 = (x+y)(x-y)$

Example: 851 = 900 – 49 = (30 + 7)(30 – 7) = (37)(23)

Difference of Cubes: $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Example: 7999 = 8000 – 1 =

$$(20-1)(20^2+(20)(1)+1^2)=(19)(400+20+1)=(19)(421)$$

Sum of Cubes: $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

Example: 1027 = 1000 + 27

$$(10+3)(10^2-(10)(3)+3^2)=(13)(100-30+9)=(13)(79)$$

Simplifying Complicated Numerical Expressions Of Large Numbers With Algebra

Look for patterns and replace common large numbers with one or two variables. Simplify the resulting algebraic expression. Replace back the original values and simplify the numerical as needed.

Example: Calculate the value of

$$\frac{(1775^2 + 8875)(1775^2 - 3550 - 3)}{5325(1775^2 + 3550 - 15)}$$

Let $n = 1775$. Now rewrite the given numerical expression, and simplify.

So we get $\dfrac{(n^2+5n)(n^2-2n-3)}{3n(n^2+2n-15)} = \dfrac{n(n+5)(n-3)(n+1)}{3n(n+5)(n-3)} = \dfrac{n+1}{3}$

Since n = 1775, $\dfrac{n+1}{3} = \dfrac{1776}{3} = 592$

Determine Specific Digits In Large Numerical By Using Patterns

Begin investigating the required calculation, concentrating on the specific digit(s) involved and any other digits that will effect the specific digit(s). A pattern or a way to simplify the problem should soon become evident.

Example 1: Find the *last two digits* of 1! + 2! + 3! + 4! + … + 100!

Start calculating factorials. The first ones are easy.

1! = 1 2! = 2 3! = 6 4! = 24 5! = 120 6! = 720 7! = 7(6!) = 5040

8! = 8(7!) = 40320 9! = 9(8!) = 362880 10!= 10(9!) = 3628800

11! = 11(10!) = 39916800

But wait! After 10! The last two digits will *always be zero*. So just add up the last two digits of what we have up to 10!.

1 + 2 + 6 + 24 + 20 + 20 + 40 + 20 + 80 = 213

Thus the last two digits of the given sum will be 13.

Determine Specific Digits In Large Numerical By Using Patterns (cont'd)

Example 2: Find the *units digit* of 2007^{2006}

Investigate units digit of the powers of the given units digit 7

$7^1 = 7$ $7^2 = 9$ $7^3 = 3$ $7^4 = 1$

$7^5 = 7$ $7^6 = 9$ $7^7 = 3$ $7^8 = 1$

Notice the pattern. The units digits repeat in cycles of 4, so $7^{2004} = 7^4 = 1$

Thus the *units digit* of 2007^{2006} will be 9

Grouping To Create A Pattern

Look for something a sequence of terms has in common, and form a grouping accordingly. Utilize the grouping to simplify.

Example 1 : Find the sum $1 + 2 + 3 + \ldots + 99 + 100$

Working the sum from the ends to the middle instead of left to right we get

$(1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$

Since there are 50 pairs of the same sum of 101,

the required sum = 50(101) = 5050

Great news if you forgot the formula! arithmetic sequence sum: $S = \frac{n}{2}(a + l)$

Example 2: Compute $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$

Grouping terms that are a difference of squares

$(100^2 - 99^2) + (98^2 - 97^2) + \ldots + (2^2 - 1^2)$ and now factoring we get

$(100 - 99)(100 + 99) + (98 - 97)(98 + 97) + \ldots + (2 - 1)(2 + 1)$ or

$1(100 + 99) + 1(98 + 97) + \ldots + 1(2 + 1) =$

$100 + 99 + 98 + \ldots + 2 + 1 = 5050$ (See above example)

Using Variables In Infinite Patterns

In general, look for the pattern of repetition and make an appropriate substitution so that the infinite nature of the expression is replaced with some closed form.

Example 1: Find the sum $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

Let $x = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$ This means $\frac{1}{4}x = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

So we can rewrite $x = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

as $x = 1 + \frac{1}{4}x$

Solving we get $4x = 4 + x$ $x = 4/3$

(Good news if you can't remember the formula for the sum of a geometric series)

$$S = \frac{a}{1 - r}$$

Example 2: Compute $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$

Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$ So we can rewrite this as $x = \sqrt{2 + x}$

Solving by squaring both sides, we get $x^2 = 2 + x$

so $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$ (reject)

Using Variables In Infinite Patterns (cont'd)

Example 3: Express $.\overline{18}$ as a fraction in simplest form

Let $x = .181818.....$ then $100x = 18.1818\ 18...$

Consider $100x - x = 18.181818... - .181818...$ or $99x = 18$

So $x = 18/99 = 2/11$

Useful short cut :

When the repeating pattern starts right after the decimal point, the equivalent (but maybe not simplest form) fraction will be one cycle of the repeating digits over as many nines.

$.\overline{1} = \frac{1}{9}$ $.\overline{73} = \frac{73}{99}$ $.\overline{1235} = \frac{1235}{9999}$ $.\overline{9} = \frac{9}{9} = 1$

When the repeating pattern starts n places to the right of the decimal point, express the non-repeating portion as usual and then add the repeating portion, as determined above, but where n zeros are included after the appropriate number of nines as a place adjustment.

$.0\overline{12} = \frac{12}{990}$ $.6\overline{213} = \frac{6}{10} + \frac{213}{9990} = \frac{5994}{9990} + \frac{213}{9990} = \frac{6207}{9990}$

Using Variables To Simplify Complex Expressions

Let a variable represent common complex expressions to simplify the problem to something familiar.

Example 1: What are the four integers for which $(x^2 + x)^2 - 18(x^2 + x) + 72 = 0$

Let y = $x^2 + x$ So the problem becomes $y^2 - 18y + 72 = 0$

Solving this new equation we get $(y-6)(y-12) = 0$

Replacing back gives us $(x^2 + x - 6)(x^2 + x - 12) = 0$

So factor $(x+3)(x-2)(x+4)(x-3) = 0$

Thus the roots are {-3, 2, -4, 3}

Example 2: Solve this system for (*a*, *b*), where *a* and *b* are real numbers:

$$a - b = 40$$

$$\sqrt{a} - \sqrt{b} = 4$$

Let $x = \sqrt{a}$ and $y = \sqrt{b}$ so the problem becomes

$$x^2 - y^2 = 40$$

$$x - y = 4$$

Now $x^2 - y^2 = (x-y)(x+y) = 40$

Notice how factoring gives us a shortcut!

But if $x - y = 4$, then $x + y = 10$

So $2x = 14$, $x = 7$, and thus $y = 3$

Replacing back, since $x = \sqrt{a} = 7$, $a = 49$ and since $y = \sqrt{b} = 3$, $b = 9$

Thus the solution is (49, 9)

Using Cases To Solve Problems

Cases are useful ways to consider problems. They may actually ***simplify***, or ***speed*** solutions, and even get you unstuck.

Using Cases To Solve Problems (cont'd)

Example 1: Solve a(a + 10) = 24

So the problem is to find factors of 24 that are 10 apart.

Theses will be 2(12) or -2(-12)

Thus a = 2 or a = -2

Example 2: Find all ordered pairs of *positive* integers (*p*, *q*) such that

$$p^2 - 3pq + 2q^2 = 21$$

One equation with two variables! So are you stuck?

No, since you can factor the left side of the equation and consider cases

$$(p - 2q)(p - q) = 21$$

Cases: (7)(3) (-7)(-3) (21)(1) (-21)(-1)

Note: Since p >0 and q >0, then $p - q > p - 2q$, so we know which factor must be the greater value in each case, making the process more efficient.

$p - 2q = 3$	$p - 2q = -7$	$p - 2q = 1$	$p - 2q = -21$
$p - q = 7$	$p - q = -3$	$p - q = 21$	$p - q = -1$

These cases solve as follows

$q = 4, p = 11$	$q = 4, p = 1$	$q = 20, p = 41$	$q = 20, p = 19$

So the solutions are { (11, 4), (1, 4), (41, 20), (19, 20) }

Example 3: Find all integral solution of x that satisfy

$$(x^2 - x - 1)^{x+2} = 1$$

Cases:

$a^o = 1$, where $a \neq 0$ or	$1^n = 1$ or	$(-1)^n = 1$, where n is even
So $x + 2 = 0$	$x^2 - x - 1 = 1$	$x^2 - x - 1 = -1$
$x = -2$	$x^2 - x - 2 = 0$	$x^2 - x = 0$
Where $x^2 - x - 1 \neq 0$	$(x - 2)(x + 1) = 0$	$x(x - 1) = 0$
And $(-2)^2 - (2) - 1 \neq 0$	x = 2 or x = -1	$x = 0$ or $x = 1$ but **n** must be ***even***
		If $x = 0$, $n = 2$, which is even
		If $x = 1$, $n = 3$, so reject this x sol.

Thus the solution is { -2, 2, -1, 0}

Factoring By Parts

Factoring can be done by parts. Look for common terms for which a factoring rule can be applied, and regroup the terms for convenience and factor. After each step, repeat the process until the entire expression is in factored form.

Example: Factor completely : $n^3 - 25n^2 - n + 25$

Grouping first two and last two $n^2(n - 25) - 1(n - 25)$

Notice common factor (n - 25) $(n - 25)(n^2 - 1)$

$(n - 25)(n - 1)(n + 1)$

Changing A Problem To An Expression Resembling A Familiar Pattern

Inspect the original problem for clues reminding you of an easier or familiar expression and determine if there is a value or operation that can change it to the desired form which can be worked and then changed back with an inverse operation or value.

For example, if c = a chosen adjustment value or expression, try one of the following

1. multiply by c, work with the result, then divide by c
2. add c, work with the result and add (-c)
3. square the expression, work with the result, then square root, being careful to recall that $\sqrt{x^2} = |x|$

Example 1 *[1983T7]

The sum $\frac{1}{1!9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!} + \frac{1}{9!1!}$ can be written in the form $\frac{2^a}{b!}$ where a and b are positive integers. Find the ordered pair (a, b).

Solution

Notice that each denominator has two factorials that would occur with numerators of 10! in combinations calculations of the form ${}_{10}C_r$, and that they all occur in the same row of Pascal's Δ. Recall also that the sum of all elements of this row would be 2^{10}.

So let x = the desired sum, and multiply it by 10!.

Let $x = \frac{1}{1!9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!} + \frac{1}{9!1!}$

So $10!(x) = \frac{10!}{1!9!} + \frac{10!}{3!7!} + \frac{10!}{5!5!} + \frac{10!}{7!3!} + \frac{10!}{9!1!}$

$= {}_{10}C_1 + {}_{10}C_3 + {}_{10}C_5 + {}_{10}C_7 + {}_{10}C_9$

Since ${}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_{10} = 2^{10}$, and since in a given row of Pascal's Δ, the sum of all ${}_nC_{odd}$ terms = sum of all ${}_nC_{even}$ terms, then

$10!(x) = {}_{10}C_1 + {}_{10}C_3 + {}_{10}C_5 + {}_{10}C_7 + {}_{10}C_9 = \frac{1}{2}$ the sum of the row $= \frac{1}{2}(2^{10})$
$= 2^9$

Now divide by (10!) to determine our original x

$x = \frac{10!(x)}{10!} = \frac{2^9}{10!}$ Thus our ordered pair is (9, 10)

Example 2

Factor completely $x^4 + 4$

Solution:

Add $4x^2$ to complete the square, but also add $-(4x^2)$ to maintain an equivalent expression.

$$\begin{aligned} x^4 + 4 &= x^4 + 4x^2 + 4 - (4x^2) = (x^2+2)^2 - (2x)^2 \\ &= ((x^2+2) + 2x)((x^2+2) - 2x) \quad \text{(by difference of squares rule)} \\ &= (x^2 + 2x + 2)(x^2 - 2x + 2) \end{aligned}$$

Example 3:

Compute $\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}$

Solution:

Let $x=\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}$ now square the expression

$$x^2=5+2\sqrt{6}+2\sqrt{(5+2\sqrt{6})(5-2\sqrt{6})}+5-2\sqrt{6}$$

So $x^2=5+2\sqrt{25-24}+5=5+2+5=12$

Now square root, so $x=\sqrt{12}=2\sqrt{3}$

Making Use of the Given Equation/Function

Example 1: [1977T2]

Find the ordered triple of real numbers (a, b, c) which satisfy

$$a(x-1)(x+1)+b(x-1)(x-3)+c(x+1)(x-3)=12x-20$$

Solution:

Notice you can use $f(1)$ to isolate and solve for c.

$f(1)=a(0)(2)+b(0)(-2)+c(2)(-2)=12(1)-20=-8$, so $-4c=-8$, and $c=2$

Similarly, use $f(-1)$ and $f(3)$ to solve for a and b

$f(-1)=a(-2)(0)+b(-2)(-4)+c(0)(-4)=12(-1)-20=-32$, so $8b=-32$, and $b=-4$

$f(3)=a(2)(4)+b(2)(0)+c(4)(0)=12(3)=12(3)-20=16$, so $8a=16$, and $a=2$

Thus our triple is (2, -4, 2)

Example 2: [2006I5]

Compute the sum of the reciprocals of the solutions to

$$x^5+5x^4-5x^3-25x^2+4x+20=0$$

Solution:

Let $y=1/x$ the reciprocal of x, so then $x=1/y$

So using the given equation: $\left(\frac{1}{y}\right)^5+5\left(\frac{1}{y}\right)^4-5\left(\frac{1}{y}\right)^3-25\left(\frac{1}{y}\right)^2+4\left(\frac{1}{y}\right)+20=0$

We get $\frac{1}{y^5}+\frac{5}{y^4}-\frac{5}{y^3}-\frac{25}{y^2}+\frac{4}{y}+20=0$ So rewriting (multiply by y^5)

We have $1+5y-5y^2-25y^3+4y^4+20y^5=0$

and it's the sum of its solutions will be $-\frac{(4)}{(20)}=-\frac{1}{5}\ or\ -0.2$

Example 3: [1992I7]

Let r be a root of $x^2+5x+7=0$. Compute $(r-1)(r+2)(r+6)(r+3)$

Solution:

Since r is a solution of the given equation, $r^2+5r+7=0$. Notice the $5r$, and group the factors accordingly before multiplying $[(r-1)(r+6)][(r+2)(r+3)]=$

$$(r^2+5r-6)(r^2+5r+6)$$

Now rewrite the numerical factors: $(r^2+5r+7-13)(r^2+5r+7-1)$

Using $r^2+5r+7=0$ we get (0 - 13)(0 - 1) =13

Lines Passing Through Lattice Points

When given a linear equation, $y = mx + b$, in order to find the lattice points that the given line passes through, find the slope of the line m, where $m = \frac{\Delta y}{\Delta x}$ and any integer solution (c,d). The line will pass through these lattice points

$(c + \Delta xk, d + \Delta yk)$ where k is an integer.

Example: Find all lattice points that the line $3x + 4y = 17$ passes through in quadrant I .

By inspection we see the line will pass through the point (3,2).
The slope of this line is $m = -3/4$
So other lattice points will be $(3 + 4k, 2 - 3k)$
$k = 0$ give the point (3, 2)
$k = 1$ gives the point $(3 + 4, 2 - 3)$ or (7, -1), but this is no longer in quadrant I
So try $k = -1$. This gives $(3 - 4, 2 + 3)$ or (-1,5), also not in quadrant I.
Thus is only lattice point is QI is (3, 2)

Graphic Solutions

Consider a graphic solution to algebraic equations or inequalities, especially if the question involves finding if a solution exists or asks for the number of solutions.

GOAL in Inequalities

Since the G O A L is to get the problem correct, remember

Greater than (>) inequalities uses "**O**r" , while
"**A**nd" is used in **L**ess than (<) inequalities

However, it never hurts to check!

Negative Cases

Don't forget a negative case when square roots and absolute values:

$a^2 = n^2$ so $a = n$ or $a = -n$

$|a| = n$ so $a = n$ or $a = -n$

$\sqrt{a^2} = |a|$

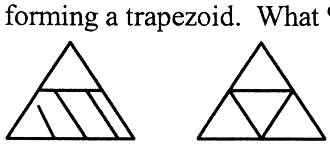
Also when solving equations by considering cases based on numerical factors, don't forget a negative case, (unless positive solutions are specified)

$ab = 5 = (5)(1) = (-5)(-1)$

Geometry Area Problems

Divide the figure into smaller congruent figures that are familiar.

Example : The midpoints of 2 sides of an equilateral Δ are joined (called mid-line), forming a trapezoid. What % of the area of the triangle is the trapezoid?



Divide into smaller congruent equilateral triangles

Area of the trapezoid is 75%
(3 out of the 4 smaller congruent triangles)

*Note: Some examples are problems used at NYSML and the year and question number are indicated.

<u>Geometry Tips</u>

1. LOOK FOR SIMILAR FIGURES, or CPCT, or RT Δs and PYTHAGOREAN TRIPLES.

2. USE GRID PAPER TO MAKE ACCURATE OR TO SCALE DRAWINGS.

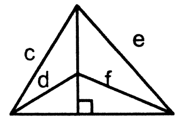
3. KNOW FORMULAS THAT CAN SAVE CALCULATING TIME:

   Area of equilateral triangle: $A = \frac{s^2\sqrt{3}}{4}$

   Area of a rhombus: $A = \frac{1}{2}(d_1)(d_2)$

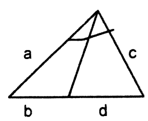
4. USING COMMON SIDES such as the common altitudes in these Δs

   $c^2 - d^2 = e^e - f^2$

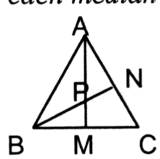


5. IF THE PROBEM INVOLVES AN ANGLE BISECTOR,

   consider the Angle Bisector Thm $\frac{a}{b} = \frac{c}{d}$



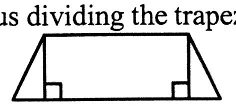
6. IF THE PROBLEM INVOLVES CONCURRENT MEDIANS IN A Δ., consider using the Centroid Thm: *The medians of a Δ intersect in a point called the centroid, which divided each median into segments in the ration of 2: 1*



   AP = 2(PM)

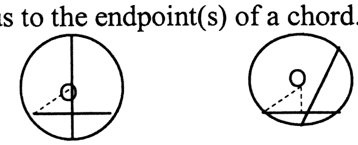
   BP = 2(PN)

7. IF THE PROBLEM INVOLVES A TRAPEZOID AND ITS HEIGHT, draw **<u>two altitudes</u>**, thus dividing the trapezoid into a rectangle and two right Δs.



   Note: If the trapezoid is and isosceles trapezoid, the two right Δs will be ≅ .

8. IF THE PROBLEM INVOLVES two chords in a circle and its radius or diameter, draw a radius to the endpoint(s) of a chord.



9. SPECIAL TRIANGLE: 13, 14, 15